Flavor charges and flavor states of mixed neutrinos

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We discuss flavor charges and states for interacting mixed neutrinos in QFT. We show that the usual Pontecorvo states are not eigenstates of the flavor charges. This implies that their use in describing the flavor neutrinos produces a violation of lepton charge conservation in the production/detection vertices. On the other hand, flavor states defined as eigenstates of the flavor charges, give the correct representation of mixed neutrinos in charged current weak interaction processes. We also present the computation of the weak interaction decays $W^+ \to e^+ + \nu_e$ and $W^+ \to e^+ + \nu_\mu$, both in the Pontecorvo formalism and in the QFT formalism. The results are shown to coincide in the relativistic limit.

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I. INTRODUCTION

Given the importance of neutrino mixing and oscillations in the context of particle physics, it is not surprising that a great deal of work has been recently devoted to some related theoretical issues. One of such aspects is the problem of the definition of flavor states, i.e. the ones describing the mixed and oscillating neutrinos. The standard treatment of flavor states in the context of Quantum Mechanics (QM) adopts the well known Pontecorvo states [1]-[13], which are certainly a good tool for capturing the main physical features of oscillating neutrinos. However, it is clear since several years that conceptual problems arise in connection with a proper definition of the flavor states. In fact, it was even stated [13] that it is impossible to construct such states and a formalism has been developed in order to avoid their use in the calculation of oscillation probabilities [14].

The root of such difficulties has been found by tackling the problem in the context of Quantum Field Theory (QFT). In this context, it has emerged [15] that field mixing is associated with inequivalent representations, i.e. the vacuum for the mass eigenstates of neutrinos turns out to be unitarily inequivalent to the vacuum for the flavor eigenstates of neutrinos. The non–perturbative vacuum structure associated with the field mixing [15]–[27] leads to a modification of flavor oscillation formulas [16, 20, 22], exhibiting new features with respect to the usual quantum mechanical ones [1]-[7]. Further developments include Lorentz invariance violation [25, 26] and neutrino mixing contribution to the dark energy of the Universe [28, 29].

This paper has two distinct, but related, aims: Firstly, in Sections II, we consider in detail the definition of the flavor charges in the canonical formalism for interacting (Dirac) neutrinos, with and without mixing. On this basis, we analyze, in Section III, the flavor states for mixed neutrinos in the QFT formalism and in the Pontecorvo formalism. We show that Pontecorvo mixed states are not eigenstates of the neutrino flavor charges and we estimate how much the leptonic charge is violated on these states.

The QFT treatment offers a natural solution to the problem of the definition of flavor states: indeed, contrarily to the Pontecorvo states, the mixed states are introduced as eigenstates of the flavor charge on the basis of the canonical definition of charges and currents from the symmetry properties of the neutrino Lagrangian [17]. Now, in order to have a realistic description of the flavor neutrinos, it is necessary to take into account the (charged current) weak interaction processes in which they are created, together with their charged lepton counterparts. In Ref.[30] the interaction at the production vertex has been incorporated in the QFT mixing formalism. There the neutrino lepton charge was computed in decay processes where neutrinos are generated. In the present paper we continue on this line by studying the charges and currents for interacting mixed neutrinos.

In the second part of the paper, Section IV, we address ourselves to our second aim, namely the computation of the weak interaction processes, such as $W^+ \to e^+ + \nu_e$ and $W^+ \to e^+ + \nu_\mu$, in the Pontecorvo formalism as well as in the QFT formalism.

The results in the Pontecorvo formalism and in the QFT one are shown to coincide in the relativistic limit or in the limit of equal mass neutrinos (in this last occurrence no mixing occurs, indeed). We also show that by using the adiabatic hypothesis, by which time integrations are taken from $t = -\infty$ to $t = +\infty$, results contrasting with the form of the weak interaction Hamiltonian are obtained both in the Pontecorvo case and the in QFT case. Then a naif conclusion could be that the Pontecorvo formalism is inconsistent and that the QFT formalism is inconsistent. However, it is not so.

We show indeed that the source of the contradiction arising in the computation with the Pontecorvo states and the QFT states is in "abusing" of the adiabatic hypothesis: since mixed neutrinos oscillates, one cannot treat them [30] as usual Lehmann-Symanzik-Zimmermann (LSZ) fields in QFT [31]. Therefore much care and wisdom is needed in order to not making confusing statements about consistency or inconsistency of the Pontecorvo formalism and of the QFT mixing formalism [32, 33].

Finally, in both formalisms the contradiction with results expected on the basis of the weak interaction Hamiltonian is absent in the realistic limit where neutrino masses can be neglected.

Similar conclusions hold for a further representation [34] of flavor states, also considered in Section IV. This last representation produces results different from the Pontecorvo ones and in contradiction with lepton charge conservation in the decay vertex. The contradiction is removed in the limit of realistic experimental conditions.

Section V is devoted to conclusions. In the Appendix, a brief summary of the vacuum structure for fermion Dirac mixing is presented.

II. FLAVOR CHARGES AND WEAK INTERACTION

In this Section, we consider the flavor charges for mixed neutrinos which enter the charged current weak interaction Lagrangian together with their corresponding charged leptons.

We discuss first the case of no mixing (one generation) and then extend our discussion to the case of mixing of two generations. Extension to three generations can also be done [22]. For simplicity and sake of shortness we do not consider it here.

A. Massive neutrinos, no mixing - one generation

We consider the decay process $W^+ \to e^+ + \nu_e$. After spontaneous symmetry breaking, the relevant terms of the Lagrangian density for charged current weak interaction are [7]

$$\mathcal{L} \;=\; \mathcal{L}_0 + \mathcal{L}_{int}\,, \tag{1}$$

where $\mathcal{L}_0$ is the free lepton Lagrangian

$$\mathcal{L}_0 \;=\; \bar{\nu}_e(x)\,(i\gamma_\mu\partial^\mu - m_{\nu_e})\,\nu_e(x) \;+\; \bar{e}(x)\,(i\gamma_\mu\partial^\mu - m_e)\,e(x)\,, \tag{2}$$

with $\bar{\nu}_e(x) = \nu_e^\dagger(x)\gamma_0$, m_e the electron mass and m_{ν_e} the (electron) neutrino mass. $\mathcal{L}_{int}$ is the charged current interaction term with the W bosons:

$$\mathcal{L}_{int} \;=\; \frac{g}{2\sqrt{2}}\left[W_\mu^+(x)\,\overline{\nu}_e(x)\,\gamma^\mu\,(1-\gamma^5)\,e(x) + h.c.\right]\,. \tag{3}$$

$\mathcal{L}$ is invariant under the transformations:

$$e(x) \to e^{i\alpha}e(x)\,, \qquad \nu_e(x) \to e^{i\alpha}\nu_e(x)\,; \tag{4}$$

from which it follows the Noether (flavor) charge

$$Q_e^{tot} \;=\; Q_{\nu_e} + Q_e\,, \tag{5}$$

$$Q_{\nu_e}(t) \;\equiv\; \int d^3\mathbf{x}\,\nu_e^\dagger(x)\;\nu_e(x)\,, \tag{6}$$

$$Q_e(t) \;\equiv\; \int d^3\mathbf{x}\,e^\dagger(x)\;e(x)\,, \tag{7}$$

where we use $x_0 \equiv t$. Since at equal time it is

$$[Q_{\nu_e}(t)\,,\,\mathcal{L}_{int}(x)] = -\,[Q_e(t)\,,\,\mathcal{L}_{int}(x)]\,, \tag{8}$$

we have $[Q_e^{tot}\,,\,\mathcal{L}_{int}(x)] = 0$, which guaranties the well known fact that the total lepton charge is conserved in (charged current) weak interaction processes. Furthermore, we have:

$$[Q_e^{tot}\,,\,\mathcal{L}_0(x)] = 0\,. \tag{9}$$

Thus, in a process described by the Lagrangian (1), the electron neutrino state $|\nu_e\rangle$ is well defined as eigenstate of the flavor charge Q_{ν_e} (6), provided, of course, its energy is not enough to produce a decay of $|\nu_e\rangle$ through the vertex (3).

B. Neutrino mixing, two generations

We now turn to the case when neutrino mixing is present and consider for simplicity only two generations. The Lagrangian is again written as

$$\mathcal{L} \;=\; \mathcal{L}_0 + \mathcal{L}_{int}\,, \tag{10}$$

where now $\mathcal{L}_0$ is the free lepton Lagrangian

$$\mathcal{L}_0 \;=\; (\bar{\nu}_e, \bar{\nu}_\mu)\,(i\gamma_\mu\partial^\mu - M_\nu)\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} \;+\; (\bar{e}, \bar{\mu})\,(i\gamma_\mu\partial^\mu - M_l)\begin{pmatrix}e\\ \mu\end{pmatrix}, \tag{11}$$

including the neutrino non-diagonal mass matrix M_ν and the mass matrix of charged leptons M_l:

$$M_\nu \;=\; \begin{pmatrix} m_{\nu_e} & m_{\nu_{e\mu}} \\ m_{\nu_{e\mu}} & m_{\nu_\mu}\end{pmatrix} \quad ; \qquad M_l \;=\; \begin{pmatrix} m_e & 0 \\ 0 & m_\mu\end{pmatrix}. \tag{12}$$

$\mathcal{L}_{int}$ is the interaction Lagrangian given by

$$\mathcal{L}_{int} \;=\; \frac{g}{2\sqrt{2}}\left[W_\mu^+(x)\,\overline{\nu}_e(x)\,\gamma^\mu\,(1-\gamma^5)\,e(x) + W_\mu^+(x)\,\overline{\nu}_\mu(x)\,\gamma^\mu\,(1-\gamma^5)\,\mu(x) + h.c.\right]. \tag{13}$$

Now, $\mathcal{L}$ is invariant under the global phase transformations:

$$e(x) \to e^{i\alpha}e(x)\,, \qquad \nu_e(x) \to e^{i\alpha}\nu_e(x)\,, \tag{14}$$

together with

$$\mu(x) \to e^{i\alpha}\mu(x)\,, \qquad \nu_\mu(x) \to e^{i\alpha}\nu_\mu(x)\,. \tag{15}$$

These are generated by

$$Q_e(t) \;=\; \int d^3\mathbf{x}\, e^\dagger(x)e(x)\,, \qquad Q_{\nu_e}(t) = \int d^3\mathbf{x}\,\nu_e^\dagger(x)\nu_e(x)\,, \tag{16}$$

$$Q_\mu(t) \;=\; \int d^3\mathbf{x}\, \mu^\dagger(x)\mu(x)\,, \qquad Q_{\nu_\mu}(t) = \int d^3\mathbf{x}\,\nu_\mu^\dagger(x)\nu_\mu(x)\,, \tag{17}$$

respectively. The invariance of the Lagrangian is then expressed by

$$[Q_l^{tot}\,,\,\mathcal{L}(x)] = 0\,, \tag{18}$$

which guarantees the conservation of total lepton number and where Q_l^{tot} is the total Noether (flavor) charge:

$$Q_l^{tot} = Q_{\nu_e}(t) + Q_{\nu_\mu}(t) + Q_e(t) + Q_\mu(t) \;=\; Q_e^{tot}(t) \;+\; Q_\mu^{tot}(t)\,, \tag{19}$$

$$Q_e^{tot}(t) = Q_{\nu_e}(t) + Q_e(t)\,, \qquad Q_\mu^{tot}(t) = Q_{\nu_\mu}(t) + Q_\mu(t)\,. \tag{20}$$

Note that the form of the flavor charges (16), (17) is the same as in the case where the mixing is absent Eqs. (6) and (7) and that the presence of the mixed neutrino mass term, i.e. of the non-diagonal mass matrix M_ν, now prevents the invariance of the Lagrangian $\mathcal{L}_0$ under the separate phase transformations (14) and (15). Indeed we have:

$$[Q_e^{tot}(t)\,,\,\mathcal{L}_0(x)] \neq 0\,, \tag{21}$$

$$[Q_\mu^{tot}(t)\,,\,\mathcal{L}_0(x)] \neq 0\,. \tag{22}$$

However, the charges Q_e^{tot} and Q_μ^{tot}, even in the presence of the mixing in the neutrino sector, still commute separately with the interaction Lagrangian $\mathcal{L}_{int}$:

$$[Q_e^{tot}(t)\,,\,\mathcal{L}_{int}(x)] = 0\,, \tag{23}$$

$$[Q_\mu^{tot}(t)\,,\,\mathcal{L}_{int}(x)] = 0. \tag{24}$$

Thus, the considerations done above about the definition of flavor neutrino states hold also in the present case: even when mixing is present, a flavor neutrino state is well defined in the production vertex as an eigenstate of the neutrino flavor charge (Q_{ν_e} for electron neutrinos, Q_{ν_μ} for muon neutrinos). In practice, such a situation is realized when, as usually it happens, the spatial extension of the neutrino source is much smaller than the neutrino oscillation length.

C. Charges for mixed neutrinos

In order to further discuss the flavor charges (and currents) for mixed particles, we now restrict ourselves to the neutrino part of the above free Lagrangian $\mathcal{L}_0$. This is justified since the interaction term in the Lagrangian (10) is diagonal in the flavor fields. The Lagrangian describing two free Dirac fields with masses m_1, m_2 and $m_1 \neq m_2$ is written as

$$\mathcal{L}_\nu(x) \;=\; \bar{\nu}_m(x) \left(i \not\partial - M_\nu^d\right) \nu_m(x) \,, \tag{25}$$

where $\nu_m^T \;=\; (\nu_1, \nu_2)$ and $M_\nu^d \;=\; diag(m_1, m_2)$. $\mathcal{L}_\nu(x)$ is invariant under global $U(1)$ phase transformations of the type $\nu_m^{'}(x) \;=\; e^{i\alpha}\nu_m(x)$. This implies the conservation of the Noether charge $Q_\nu \;=\; \int I^0(x) d^3\mathbf{x}$ (with $I^\mu(x) = \bar{\nu}_m(x)\gamma^\mu\nu_m(x)$) which is indeed the total charge of the system, i.e. the total lepton number of neutrinos.

Consider now the global $SU(2)$ transformation [17]:

$$\nu_m'(x) = e^{i\alpha_j \cdot \tau_j} \nu_m(x) \qquad\qquad j = 1, 2, 3. \tag{26}$$

with α_j real constants, $\tau_j = \sigma_j/2$ with σ_j being the Pauli matrices.

$\mathcal{L}_\nu$ is not invariant under the transformations (26) since $m_1 \neq m_2$. By use of the equations of motion, we obtain

$$\delta\mathcal{L}_\nu = i\alpha_j \bar{\nu}_m(x) \left[\tau_j ,\; M_\nu^d\right] \nu_m(x) = -\alpha_j \partial_\mu J^\mu_{m,j}(x) \,, \tag{27}$$

where the currents are:

$$J^\mu_{m,j}(x) = \bar{\nu}_m(x)\, \gamma^\mu\, \tau_j\, \nu_m(x) \,, \qquad\qquad j = 1, 2, 3. \tag{28}$$

The related charges

$$Q_{m,j}(t) = \int d^3\mathbf{x}\, J^0_{m,j}(x) \,, \tag{29}$$

satisfy the $su(2)$ algebra: $[Q_{m,i}(t), Q_{m,j}(t)] = i\varepsilon_{ijk} Q_{m,k}(t)$. The Casimir operator is proportional to the total (conserved) charge: $Q_{m,0} = \frac{1}{2} Q_\nu$ and also $Q_{m,3}$ is conserved, due to the fact that M_ν^d is diagonal. This implies that the charges for ν_1 and ν_2 are separately conserved. The $U(1)$ Noether charges associated with ν_1 and ν_2 can be then expressed as

$$Q_{\nu_1} \;\equiv\; \frac{1}{2} Q_\nu \;+\; Q_{m,3} \,; \qquad\qquad Q_{\nu_2} \;\equiv\; \frac{1}{2} Q_\nu \;-\; Q_{m,3} \,. \tag{30}$$

$$Q_{\nu_i} \;=\; \int d^3\mathbf{x}\, \nu_i^\dagger(x)\, \nu_i(x) \,, \tag{31}$$

with Q_ν total (conserved) charge and $i = 1, 2$.

Let us now consider the Lagrangian $\mathcal{L}_\nu(x)$ written in the flavor basis

$$\mathcal{L}_\nu(x) \;=\; \bar{\nu}_f(x) \left(i \not\partial - M_\nu\right) \nu_f(x) \,, \tag{32}$$

where $\nu_f^T = (\nu_e,\, \nu_\mu)$. The variation of the Lagrangian (32) under the $SU(2)$ transformation:

$$\nu_f'(x) = e^{i\alpha_j \cdot \tau_j} \nu_f(x) \qquad\qquad j = 1, 2, 3 \,, \tag{33}$$

is given by

$$\delta\mathcal{L}_\nu(x) = i\alpha_j \bar{\nu}_f(x) \left[\tau_j, M_\nu\right] \nu_f(x) = -\alpha_j \partial_\mu J^\mu_{f,j}(x) \,, \tag{34}$$

where

$$J^\mu_{f,j}(x) = \bar{\nu}_f(x)\, \gamma^\mu\, \tau_j\, \nu_f(x) \,, \qquad j = 1, 2, 3 \,. \tag{35}$$

Again, the charges

$$Q_{f,j}(t) = \int d^3\mathbf{x}\, J^0_{f,j}(x) \tag{36}$$

close the $su(2)$ algebra, however, because of the off-diagonal (mixing) terms in M_ν, $Q_{f,3}(t)$ is time dependent. This implies an exchange of charge between ν_e and ν_μ, resulting in the phenomenon of neutrino oscillations. The (time dependent) flavor charges for mixed fields are then defined as [17]:

$$Q_{\nu_e}(t) = \frac{1}{2}Q_\nu + Q_{f,3}(t) \ , \qquad Q_{\nu_\mu}(t) = \frac{1}{2}Q_\nu - Q_{f,3}(t) \ , \tag{37}$$

$$Q_{\nu_\sigma}(t) \ = \ \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\ \nu_\sigma(x) \ , \tag{38}$$

where $\sigma = e, \mu$ and $Q_{\nu_e}(t) + Q_{\nu_\mu}(t) = Q_\nu$.

The flavor charges (38) coincide with those found in Eqs.(16) and (17) when the interaction is switched off. The $SU(2)$ group structure above discussed is important since it relates the flavor charges and the mixing generator (see the Appendix).

III. FLAVOR STATES FOR MIXED NEUTRINOS

We now define the flavor states as eigenstates of the flavor charges Q_{ν_e} and Q_{ν_μ}. Till now our considerations have been essentially classical. In order to define the eigenstates of the above charges, we quantize the fields with definite masses as usual (see Appendix). Then the normal ordered charge operators for free neutrinos ν_1, ν_2 become:

$$: Q_{\nu_i} :\equiv \int d^3\mathbf{x}\ : \nu_i^\dagger(x)\ \nu_i(x) := \sum_r \int d^3\mathbf{k}\ \left(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i} \ - \ \beta^{r\dagger}_{-\mathbf{k},i}\beta^r_{-\mathbf{k},i}\right) , \tag{39}$$

where $i = 1, 2$ and $: .. :$ denotes normal ordering with respect to the vacuum $|0\rangle_{1,2}$. The neutrino states with definite masses defined as

$$|\nu^r_{\mathbf{k},i}\rangle = \alpha^{r\dagger}_{\mathbf{k},i}|0\rangle_{1,2}, \qquad i = 1, 2, \tag{40}$$

are then eigenstates of Q_{ν_1} and Q_{ν_2}, which can be identified with the lepton charges of neutrinos in the absence of mixing.

The situation is more delicate when mixing is present. In such a case, the flavor neutrino states have to be defined as the eigenstates of the flavor charges $Q_{\nu_\sigma}(t)$ (at a given time). The relation between the flavor charges in the presence of mixing and those in the absence of mixing is:

$$Q_{\nu_e}(t) \ = \ \cos^2\theta\ Q_{\nu_1} + \sin^2\theta\ Q_{\nu_2} + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger(x)\nu_2(x) + \nu_2^\dagger(x)\nu_1(x)\right] \ , \tag{41}$$

$$Q_{\nu_\mu}(t) \ = \ \sin^2\theta\ Q_{\nu_1} + \cos^2\theta\ Q_{\nu_2} - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger(x)\nu_2(x) + \nu_2^\dagger(x)\nu_1(x)\right] \ . \tag{42}$$

Notice that the last term in these expressions is proportional to the charge $Q_{m,1}$ defined above (cf. Eq.(29)). The presence of such a term forbids the construction of eigenstates of the $Q_{\nu_\sigma}(t)$ in the Hilbert space $\mathcal{H}_{1,2}$. This fact, as well as the orthogonality (unitary inequivalence [15], see the Appendix) of the states $|0\rangle_{e,\mu}$ and $|0\rangle_{1,2}$, is a rigorous, necessary consequence of the fact the neutrinos are relativistic quantum fields. One has to live with it. Objections to this fact are mathematically and physically meaningless.

The normal ordered flavor charge operators for mixed neutrinos are then written as

$$\begin{aligned} :: Q_{\nu_\sigma}(t) :: \ &\equiv \ \int d^3\mathbf{x}\ :: \nu_\sigma^\dagger(x)\ \nu_\sigma(x) :: \\ &= \ \sum_r \int d^3\mathbf{k}\ \left(\alpha^{r\dagger}_{\mathbf{k},\nu_\sigma}(t)\alpha^r_{\mathbf{k},\nu_\sigma}(t) \ - \ \beta^{r\dagger}_{-\mathbf{k},\nu_\sigma}(t)\beta^r_{-\mathbf{k},\nu_\sigma}(t)\right) \end{aligned} \tag{43}$$

where $\sigma = e, \mu$, and $:: ... ::$ denotes normal ordering with respect to $|0\rangle_{e,\mu}$. Thus, the flavor charges are diagonal in the flavor annihilation/creation operators constructed by means of the mixing generator presented in the Appendix. The definition of the normal ordering $:: ... ::$ for any operator A, is the usual one:

$$:: A :: \equiv \ A \ - \ {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu} \, . \tag{44}$$

Note that $:: Q_{\nu_\sigma}(t) :: = \ G^{-1}_\theta(t) : Q_{\nu_j} : G_\theta(t)$, with $(\sigma, j) = (e, 1), (\mu, 2)$, and

$$:: Q_\nu :: = :: Q_{\nu_e}(t) :: \ + \ :: Q_{\nu_\mu}(t) :: = : Q_{\nu_1} : \ + \ : Q_{\nu_2} : = : Q_\nu : \ . \tag{45}$$

The flavor states are defined as eigenstates of the flavor charges Q_{ν_σ} at a reference time $t = 0$:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\nu_\sigma}(0)|0(0)\rangle_{e,\mu}, \qquad \sigma = e, \mu \tag{46}$$

and similar ones for antiparticles. We have

$$:: Q_{\nu_e}(0) :: |\nu^r_{\mathbf{k},e}\rangle \;=\; |\nu^r_{\mathbf{k},e}\rangle\,, \qquad\qquad :: Q_{\nu_\mu}(0) :: |\nu^r_{\mathbf{k},\mu}\rangle \;=\; |\nu^r_{\mathbf{k},\mu}\rangle \tag{47}$$

$$:: Q_{\nu_e}(0) ::\; |\nu^r_{\mathbf{k},\mu}\rangle \;=\; 0 \;=\; :: Q_{\nu_\mu}(0) ::\; |\nu^r_{\mathbf{k},e}\rangle\,, \qquad :: Q_{\nu_\sigma}(0) ::\; |0\rangle_{e,\mu} \;=\; 0. \tag{48}$$

The explicit form of the flavor states $|\nu^r_{\mathbf{k},e}\rangle$ and $|\nu^r_{\mathbf{k},\mu}\rangle$ at time $t = 0$ is given in the Appendix.

A. Flavor charges and Pontecorvo states

The Pontecorvo states [1]-[12]

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle\,, \tag{49}$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\,, \tag{50}$$

are clearly *not* eigenstates of the flavor charges [30] as can be seen from Eqs.(41) and (42).

In order to estimate how much the lepton charge is violated in the usual quantum mechanical states, we consider the expectation values of the flavor charges on the Pontecorvo states. We obtain, for the electron neutrino charge:

$${}_P\langle\nu^r_{\mathbf{k},e}| :: Q_{\nu_e}(0) :: |\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\sin^2\theta\cos^2\theta + \sum_r \int d^3\mathbf{k}\,, \tag{51}$$

and

$${}_{1,2}\langle 0| :: Q_{\nu_e}(0) :: |0\rangle_{1,2} = \sum_r \int d^3\mathbf{k}\,. \tag{52}$$

The infinities in Eqs.(51) and (52) may be removed by considering the expectation values of $: Q_{\nu_\sigma}(t) :$, i.e. the normal ordered flavor charges with respect to the mass vacuum $|0\rangle_{1,2}$. Then,

$${}_{1,2}\langle 0| : Q_{\nu_e}(0) : |0\rangle_{1,2} = 0\,. \tag{53}$$

However,

$${}_P\langle\nu^r_{\mathbf{k},e}| : Q_{\nu_e}(0) : |\nu^r_{\mathbf{k},e}\rangle_P = \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\sin^2\theta\cos^2\theta < 1, \quad \forall\theta \neq 0, \quad m_1 \neq m_2, \quad \mathbf{k} \neq 0, \tag{54}$$

which is still not what one wants. Moreover, the quantities

$${}_{1,2}\langle 0|(: Q_{\nu_e}(0) :)^2|0\rangle_{1,2} = 4\sin^2\theta\cos^2\theta \int d^3\mathbf{k}|V_{\mathbf{k}}|^2, \tag{55}$$

$${}_P\langle\nu^r_{\mathbf{k},e}|(: Q_{\nu_e}(0) :)^2|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^6\theta + \sin^6\theta + \sin^2\theta\cos^2\theta \left[2|U_{\mathbf{k}}| + |U_{\mathbf{k}}|^2 + 4\int d^3\mathbf{k}|V_{\mathbf{k}}|^2\right], \tag{56}$$

are both infinite, making the corresponding quantum fluctuations divergent. This confirms that the states (49) and (50) are not eigenstates of the flavor charge. The correct flavor states describing the neutrino oscillations must be those defined in Eqs. (46).

We also remark that in the standard QM treatment [1]-[12], the Pontecorvo states (49)-(50) are usually assumed to be produced, together with the respective charged (anti-)leptons, in a charged current weak interaction process. However, as shown, such states are not eigenstates of the (neutrino) lepton charges. Using such states to describe the neutrino production processes causes a *violation of lepton charge conservation*, both in the production and in the detection vertices.[1] This is in contradiction with the form of the weak interaction Hamiltonian.

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[1] In presence of mixing, the lepton charge (for a given family) is violated during time evolution (flavor oscillations), whereas the form of the weak interaction compels the lepton number to be conserved in a charged current vertex.

To be more specific, let us define [35] (cf. Eq.(54)):

$$A_0 \equiv {}_P\langle \nu^r_{\mathbf{k},e}| : Q_{\nu_e}(0) : |\nu^r_{\mathbf{k},e}\rangle_P < 1\,, \tag{57}$$

$$1 - A_0 \equiv {}_P\langle \nu^r_{\mathbf{k},e}| : Q_{\nu_\mu}(0) : |\nu^r_{\mathbf{k},e}\rangle_P = 2\sin^2\theta\cos^2\theta(1-|U_{\mathbf{k}}|) > 0, \tag{58}$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$. Consider then an ideal experiment in which neutrinos are created and detected by means of some charged weak interaction process. In the experiment one measures the number of charged leptons, say (anti-)electrons, both in the source and in the detector. Denoting with N_e^S such a number at the neutrino source and with $N_e^D(t)$ the number at the detector, in the usual treatment one assumes that $N_{\nu_e}^S = N_e^S$ and $N_{\nu_e}^D(t) = N_e^D(t)$, where $N_{\nu_e}^S$ are the neutrinos produced in the source and $N_{\nu_e}^D(t)$ are those detected. Then, according to the usual Pontecorvo formulas it is:

$$\frac{N_e^D(t)}{N_e^S} = \frac{N_{\nu_e}^D(t)}{N_{\nu_e}^S} = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta\omega}{2}t\right) = 1 - P(t) \tag{59}$$

$$\frac{N_\mu^D(t)}{N_e^S} = \frac{N_{\nu_\mu}^D(t)}{N_{\nu_e}^S} = \sin^2 2\theta \sin^2\left(\frac{\Delta\omega}{2}t\right) = P(t)\,. \tag{60}$$

We use now these formulas together with the fact that the Pontecorvo states violate the lepton charge, as shown in Eqs.(57), (58). If we assume that the source produces the Pontecorvo states (49)-(50), the conservation of leptonic charge both in the production and in the detection vertices, required by the form of the weak interaction, implies that only a fraction of the produced electron neutrinos is accompanied by an anti-electron: we denote these quantities with a tilde. We have then:

$$\tilde{N}_e^S = A_0 N_{\nu_e}^S\;, \tag{61}$$

$$\tilde{N}_e^D(t) = A_0 N_{\nu_e}^D(t) + (1-A_0) N_{\nu_\mu}^D(t)\,, \tag{62}$$

and the oscillation formula becomes

$$\frac{\tilde{N}_e^D(t)}{\tilde{N}_e^S} = \frac{A_0 N_{\nu_e}^D(t) + (1-A_0) N_{\nu_\mu}^D(t)}{A_0 N_{\nu_e}^S} = 1 - \frac{2A_0 - 1}{A_0} P(t)\,. \tag{63}$$

Eq.(63) is clearly different from the Pontecorvo formula (59). This inconsistency is removed in the relativistic limit. Eqs.(59) and (60) are indeed recovered in the limit for $|\mathbf{k}| \gg \sqrt{m_1 m_2}$ where $|U_{\mathbf{k}}| \longrightarrow 1$ and $A_0 = 1$ (cf. Eq.(57)). The exact oscillation formulas, obtained by using QFT flavor charges and flavor states, are given in the Appendix (see also Refs. [15, 22]).

IV. AMPLITUDE OF WEAK INTERACTION PROCESSES

In this Section we compute the amplitude of decay processes $W^+ \to e^+ + \nu_e$ and $W^+ \to e^+ + \nu_\mu$ by using the QFT flavor states Eq.(46) and then by using the Pontecorvo mixed states Eqs.(49), (50). In both cases we use the adiabatic hypothesis as done in Ref. [32]. We find that the two cases give coinciding results in the limit of negligible neutrino masses (as in the relativistic limit). In both cases, however, we obtain results contradicting the expected ones on the basis of the weak interaction Hamiltonian. However, one should consider that, in the Pontecorvo formalism and in the QFT one, these results are originated from the use of the adiabatic hypothesis. Much care is in fact required in applying such an hypothesis to states, such as oscillating states, which are not LSZ states [30].

We perform the computation also for the flavor states in the representation introduced in Ref. [34] and obtain similar results.

Our discussion thus clarify some confusing statements recently appeared in some papers [32, 33]. As shown here, these criticisms, involving also the Pontecorvo formalism which leads to results similar to the QFT ones, do not have any mathematical and physical justification.

A. Amplitude of weak interaction processes in QFT

It is useful to perform in detail the calculations for the processes $W^+ \to e^+ + \nu_e$ and $W^+ \to e^+ + \nu_\mu$ considered in Ref.[32].

1. Decay $W^+ \to e^+ + \nu_e$

We consider neutrinos produced through charge current process, such as

$$W^+ \to e^+ + \nu_e. \tag{64}$$

The Hamiltonian responsible for this decay is [7]

$$H_{int}(x) = -\frac{g}{\sqrt{2}} W_\mu^+(x) J_W^{\mu+}(x) = -\frac{g}{2\sqrt{2}} W_\mu^+(x)\overline{\nu}_e(x)\gamma^\mu(1-\gamma^5)e(x). \tag{65}$$

where $W^+(x)$, $e(x)$, $\nu_e(x)$ are the fields of the boson W^+, the electron and the flavor (electron) neutrino, respectively.
Assuming that the W^+ decay process (64) takes place at time $t = x_I^0$, at the first order of perturbation theory, the amplitude of the decay (64) is

$$\begin{aligned} A_{W^+\to e^+ + \nu_e} &= \langle \nu^r_{\mathbf{k},e}(x_I^0), e^s_{\mathbf{q}}| \left[-i \int_{x_{in}^0}^{x_{out}^0} d^4x \, H_{int}(x) \right] |W^+_{\mathbf{p},\lambda}\rangle \\ &= {}_W\langle 0| \, \langle \nu^r_{\mathbf{k},e}(x_I^0)| \, \langle e^s_{\mathbf{q}}| \left\{ \frac{i\,g}{2\sqrt{2}} \int_{x_{in}^0}^{x_{out}^0} d^4x \, \left[W_\mu^+(x)\overline{\nu}_e(x)\gamma^\mu(1-\gamma^5)e(x) \right] \right\} |W^+_{\mathbf{p},\lambda}\rangle \, |0\rangle_e \, |0(x_I^0)\rangle_{\nu_e}. \end{aligned} \tag{66}$$

Being

$$\begin{aligned} {}_W\langle 0|W_\mu^+(x)|W^+_{\mathbf{p},\lambda}\rangle &= \frac{1}{(2\pi)^{3/2}} \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \, e^{i(\mathbf{p}\cdot\mathbf{x} - \omega_p x^0)}, \qquad (67) \\ \langle e^s_{\mathbf{q}}|e(x)|0\rangle_e &= \frac{1}{(2\pi)^{3/2}} \, v^s_{\mathbf{q},e} \, e^{-i(\mathbf{q}\cdot\mathbf{x} - \omega_q x^0)}, \qquad (68) \\ \langle \nu^r_{\mathbf{k},e}(x_I^0)|\overline{\nu}_e(x)|0(x_I^0)\rangle_{\nu_e} &= \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \Big\{ \overline{u}^r_{\mathbf{k},1} \left[\cos^2\theta \, e^{i\omega_{k,1}(x^0-x_I^0)} + \sin^2\theta \left(|U_{\mathbf{k}}|^2 \, e^{i\omega_{k,2}(x^0-x_I^0)} + |V_{\mathbf{k}}|^2 \, e^{-i\omega_{k,2}(x^0-x_I^0)} \right) \right] \\ &+ \varepsilon^r \, |U_{\mathbf{k}}||V_{\mathbf{k}}| \, \overline{v}^r_{-\mathbf{k},1} \sin^2\theta \left[e^{-i\omega_{k,2}(x^0-x_I^0)} - e^{i\omega_{k,2}(x^0-x_I^0)} \right] \Big\}, \qquad (69) \end{aligned}$$

and assuming that $x_I^0 = 0$, Eq.(66) becomes

$$\begin{aligned} A_{W^+\to e^+ + \nu_e} &= \frac{i\,g}{2\sqrt{2}(2\pi)^{3/2}} \, \delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}) \int_{x_{in}^0}^{x_{out}^0} dx^0 \, \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \Big\{ \overline{u}^r_{\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e} \Big[\cos^2\theta \, e^{-i(\omega_p-\omega_q-\omega_{k,1})x^0} \\ &+ \sin^2\theta \left(|U_{\mathbf{k}}|^2 \, e^{-i(\omega_p-\omega_q-\omega_{k,2})x^0} + |V_{\mathbf{k}}|^2 \, e^{-i(\omega_p-\omega_q+\omega_{k,2})x^0} \right) \Big] \\ &+ \varepsilon^r \, |U_{\mathbf{k}}||V_{\mathbf{k}}| \, \overline{v}^r_{-\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e} \sin^2\theta \left[e^{-i(\omega_p-\omega_q+\omega_{k,2})x^0} - e^{-i(\omega_p-\omega_q-\omega_{k,2})x^0} \right] \Big\}. \end{aligned} \tag{70}$$

In the scattering theory for potential of finite range, it is assumed that, as $x^0_{in} \to -\infty$ and $x^0_{out} \to \infty$, the interaction Hamiltonian $H_{int}(x)$ can be switched off adiabatically and that initial and final states can be represented by eigenstates of the free Hamiltonian. In the present case, and in general in the decay processes where mixed neutrinos are produced, the adiabatic hypothesis cannot be blindly applied as done in Ref.[32]. Indeed, the flavor neutrino field operators do not have the mathematical characterization necessary in order to be defined as asymptotic field operators acting on the massive neutrino vacuum (they cannot be represented in the L^2 space of square integrable functions). Moreover, the flavor states $|\nu^r_{\mathbf{k},\sigma}\rangle$ are not eigenstates of the free Hamiltonian. Therefore much care is needed in order to avoid including neutrino oscillations effects into the decay amplitude. Thus the integration limits in Eq.(70) must be chosen so that the time interval $\Delta t = x^0_{out} - x^0_{in}$ is much smaller than the characteristic neutrino oscillation time. However, let us follow Ref.[32] and apply the adiabatic hypothesis: $x^0_{in} \to -\infty$ and $x^0_{out} \to \infty$. We have

$$\begin{aligned} A_{W^+\to e^+ + \nu_e} &= \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}} \, \delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}) \, \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \Big\{ \overline{u}^r_{\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e} \Big[\cos^2\theta \, \delta(\omega_p-\omega_q-\omega_{k,1}) \\ &+ \sin^2\theta \left(|U_{\mathbf{k}}|^2 \, \delta(\omega_p-\omega_q-\omega_{k,2}) + |V_{\mathbf{k}}|^2 \, \delta(\omega_p-\omega_q+\omega_{k,2}) \right) \Big] \\ &+ \varepsilon^r \, |U_{\mathbf{k}}||V_{\mathbf{k}}| \, \overline{v}^r_{-\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e} \sin^2\theta \left[\delta(\omega_p-\omega_q+\omega_{k,2}) - \delta(\omega_p-\omega_q-\omega_{k,2}) \right] \Big\}. \end{aligned} \tag{71}$$

We remark that the antineutrino contributions to the vacuum condensate are accounted for in the $|V_{\mathbf{k}}|$ terms in Eq.(71). This same equation is in agreement with the final result, Eq.(3.9), of Ref.[32].

At this point, however, the computation is not complete and one needs to go farther in order to disclose the meaning of such a result. We use then the relations (A16) and (A17) in the Appendix and Eq.(71) becomes

$$\begin{aligned} A_{W^+\to e^+ + \nu_e} &= \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}\Big\{\cos^2\theta\,\overline{u}^r_{\mathbf{k},1}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q-\omega_{k,1}) \\ &+\ \sin^2\theta\,\Big[|U_{\mathbf{k}}|\,\overline{u}^r_{\mathbf{k},2}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q-\omega_{k,2}) + \varepsilon^r\,|V_{\mathbf{k}}|\,\overline{v}^r_{-\mathbf{k},2}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q+\omega_{k,2})\Big]\Big\}. \end{aligned} \tag{72}$$

Since in the rest frame of the W^+ boson, the momentum of neutrinos is of the order of $m_W/2 \approx 40\,GeV$, which is much larger than the neutrino masses, then $\omega_{k,1} \simeq \omega_{k,2}$, $|U_{\mathbf{k}}| \to 1$, $|V_{\mathbf{k}}| \to 0$ and the amplitude $A_{W^+\to e^+ + \nu_e}$ goes to the standard amplitude of decay for massless neutrinos.

Therefore, in the realistic case, the W^+ boson decay rate, calculated using the flavor neutrino state (46) practically coincides with the standard W^+ boson decay rate calculated assuming massless neutrinos. Similar conclusion holds for all weak processes.

2. *Decay $W^+ \to e^+ + \nu_\mu$*

Following Ref.[32], we also consider the process $W^+ \to e^+ + \nu_\mu$. By using the Hamiltonian (65), we get

$$\begin{aligned} A_{W^+\to e^+ + \nu_\mu} &= \langle \nu^r_{\mathbf{k},\mu}(x^0_I), e^s_{\mathbf{q}}|\left[-i\int_{x^0_{in}}^{x^0_{out}} d^4x\, H_{int}(x)\right]|W^+_{\mathbf{p},\lambda}\rangle \\ &= {}_W\langle 0|\,\langle \nu^r_{\mathbf{k},\mu}(x^0_I)|\,\langle e^s_{\mathbf{q}}|\left\{\frac{i\,g}{2\sqrt{2}}\int_{x^0_{in}}^{x^0_{out}} d^4x\,\left[W^+_\mu(x)\overline{\nu}_e(x)\gamma^\mu(1-\gamma^5)e(x)\right]\right\}|W^+_{\mathbf{p},\lambda}\rangle\,|0\rangle_e\,|0(x^0_I)\rangle_{\nu_e}. \end{aligned} \tag{73}$$

Being

$$\begin{aligned} \langle \nu^r_{\mathbf{k},\mu}(x^0_I)|\overline{\nu}_e(x)|0(x^0_I)\rangle_{\nu_e} &= \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\sin\theta\cos\theta\Big\{|U_{\mathbf{k}}|\,\overline{u}^r_{\mathbf{k},1}\,\left[e^{i\omega_{k,2}(x^0-x^0_I)} - e^{i\omega_{k,1}(x^0-x^0_I)}\right] \\ &+\ \varepsilon^r\,|V_{\mathbf{k}}|\,\overline{v}^r_{-\mathbf{k},1}\left[-e^{i\omega_{k,2}(x^0-x^0_I)} + e^{-i\omega_{k,1}(x^0-x^0_I)}\right]\Big\}, \end{aligned} \tag{74}$$

and assuming that $x^0_I = 0$, Eq.(73) becomes

$$\begin{aligned} A_{W^+\to e^+ + \nu_\mu} &= \frac{i\,g}{2\sqrt{2}(2\pi)^{3/2}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\,\sin\theta\cos\theta\int_{x^0_{in}}^{x^0_{out}} dx^0\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}\Big\{|U_{\mathbf{k}}|\,\overline{u}^r_{\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e}\,\Big[e^{-i(\omega_p-\omega_q-\omega_{k,2})x^0} \\ &-\ e^{-i(\omega_p-\omega_q-\omega_{k,1})x^0}\Big] + \varepsilon^r\,|V_{\mathbf{k}}|\,\overline{v}^r_{-\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e}\left[-e^{-i(\omega_p-\omega_q-\omega_{k,2})x^0} + e^{-i(\omega_p-\omega_q+\omega_{k,1})x^0}\right]\Big\}. \end{aligned} \tag{75}$$

When the adiabatic hypothesis is applied and $x^0_{in} \to -\infty$ and $x^0_{out} \to \infty$, we have

$$\begin{aligned} A_{W^+\to e^+ + \nu_\mu} &= \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\,\sin\theta\cos\theta\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}\Big\{|U_{\mathbf{k}}|\,\overline{u}^r_{\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e}\Big[\,\delta(\omega_p-\omega_q-\omega_{k,2}) \\ &-\ \delta(\omega_p-\omega_q-\omega_{k,1})\Big] + \varepsilon^r\,|V_{\mathbf{k}}|\,\overline{v}^r_{-\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e}\left[-\delta(\omega_p-\omega_q-\omega_{k,2}) + \delta(\omega_p-\omega_q+\omega_{k,1})\right]\Big\}. \end{aligned} \tag{76}$$

Again, this reproduces the final result, Eq.(3.13), of Ref. [32]. The analysis needs, however, to be completed and we thus consider Eqs.(A16) and (A17). The amplitude $A_{W^+\to e^+ + \nu_\mu}$ then becomes

$$\begin{aligned} A_{W^+\to e^+ + \nu_\mu} &= \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}\,\sin\theta\cos\theta\,\Big[\overline{u}^r_{\mathbf{k},2}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q-\omega_{k,2}) \\ &-\ \overline{u}^r_{\mathbf{k},1}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q-\omega_{k,1}) + \varepsilon^r\,|V_{\mathbf{k}}|\,\overline{v}^r_{-\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q+\omega_{k,1})\Big], \end{aligned} \tag{77}$$

which is not zero, in contrast with the conservation of lepton charge at the vertex predicted by the weak interaction Hamiltonian. This is the price we pay for applying the adiabatic hypothesis. However, as in the previous case, since the momentum of neutrinos produced is much larger than the neutrino masses, we can set $\omega_{k,1} \simeq \omega_{k,2}$, $u^r_{\mathbf{k},1} \simeq u^r_{\mathbf{k},2}$, $|V_{\mathbf{k}}| \to 0$. Then the amplitude $A_{W^+ \to e^+ + \nu_\mu}$ goes to zero.

The above considerations hold for all the different neutrino production processes. The conclusion is that the reactions violating the conservation of total leptonic number are strongly inhibited, as it should be.

The lesson we learn from the cases 1. and 2. above considered is that only when the flavor states can be approximated with eigenstates of the free Hamiltonian the adiabatic hypothesis can be applied. In general, the energies and momenta of the particles that participate to the neutrino production process are much larger than the neutrino masses and the flavor states practically coincide with the eigenstates of the free Hamiltonian.

B. Amplitude of weak interaction processes with Pontecorvo states

The above results are better understood when compared with the decay amplitudes for same processes by utilizing the Pontecorvo states [36].

1. *Decay* $W^+ \to e^+ + \nu_e$

Eq.(69) is replaced by

$$ {}_P\langle \nu^r_{\mathbf{k},e} | \overline{\nu}_e(x) | 0 \rangle_{1,2} = \cos\theta \, \langle \nu^r_{\mathbf{k},1} | \, \overline{\nu}_e(x) \, | 0 \rangle_{1,2} + \sin\theta \, \langle \nu^r_{\mathbf{k},2} | \, \overline{\nu}_e(x) \, | 0 \rangle_{1,2} \, . \tag{78} $$

By using Eq.(A1), Eq.(78) becomes

$$ {}_P\langle \nu^r_{\mathbf{k},e} | \overline{\nu}_e(x) | 0 \rangle_{1,2} = \frac{\cos^2\theta}{(2\pi)^{3/2}} \, \overline{u}^r_{\mathbf{k},1} \, e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{k,1}\, x^0)} \; + \; \frac{\sin^2\theta}{(2\pi)^{3/2}} \, \overline{u}^r_{\mathbf{k},2} \, e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{k,2}\, x^0)}, \tag{79} $$

and the amplitude $A_{W^+ \to e^+ + \nu_e}$ is

$$ \begin{aligned} A_{W^+ \to e^+ + \nu_e} \; &= \; \frac{i\, g}{2\sqrt{2}(2\pi)^{1/2}} \, \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \, \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \Big[\cos^2\theta \, \overline{u}^r_{\mathbf{k},1} \, \gamma^\mu (1 - \gamma^5) \, v^s_{\mathbf{q},e} \, \delta(\omega_p - \omega_q - \omega_{k,1}) \\ &+ \; \sin^2\theta \, \overline{u}^r_{\mathbf{k},2} \, \gamma^\mu (1 - \gamma^5) \, v^s_{\mathbf{q},e} \, \delta(\omega_p - \omega_q - \omega_{k,2}) \Big]. \end{aligned} \tag{80} $$

Since the calculations was performed by using the Hilbert space for massive neutrinos, the flavor vacuum effects are since the beginning excluded from Eq.(80).

The Pontecorvo result Eq.(80) and the QFT result Eq.(72) are recognized to coincide in the limit $|V_{\mathbf{k}}| \to 0$, $|U_{\mathbf{k}}| \to 1$, and in the approximation $\omega_{k,1} \simeq \omega_{k,2}$, $u^r_{\mathbf{k},1} \simeq u^r_{\mathbf{k},2}$.

2. *Decay* $W^+ \to e^+ + \nu_\mu$

Eq.(74) is replaced by

$$ {}_P\langle \nu^r_{\mathbf{k},\mu} | \overline{\nu}_e(x) | 0 \rangle_{1,2} = -\sin\theta \, \langle \nu^r_{\mathbf{k},1} | \, \overline{\nu}_e(x) \, | 0 \rangle_{1,2} + \cos\theta \, \langle \nu^r_{\mathbf{k},2} | \, \overline{\nu}_e(x) \, | 0 \rangle_{1,2} \tag{81} $$

By using Eq.(A1), Eq.(81) becomes

$$ {}_P\langle \nu^r_{\mathbf{k},\mu} | \overline{\nu}_e(x) | 0 \rangle_{1,2} = \frac{\sin\theta\cos\theta}{(2\pi)^{3/2}} \, \Big[\overline{u}^r_{\mathbf{k},2} \, e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{k,2}\, x^0)} \; - \; \overline{u}^r_{\mathbf{k},1} \, e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{k,1}\, x^0)} \Big], \tag{82} $$

and the amplitude $A_{W^+ \to e^+ + \nu_\mu}$ is

$$ \begin{aligned} A_{W^+ \to e^+ + \nu_\mu} \; &= \; \frac{i\, g}{2\sqrt{2}(2\pi)^{1/2}} \, \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \, \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \, \sin\theta\cos\theta \, \Big[\overline{u}^r_{\mathbf{k},2} \, \gamma^\mu (1 - \gamma^5) \, v^s_{\mathbf{q},e} \, \delta(\omega_p - \omega_q - \omega_{k,2}) \\ &- \; \overline{u}^r_{\mathbf{k},1} \, \gamma^\mu (1 - \gamma^5) \, v^s_{\mathbf{q},e} \, \delta(\omega_p - \omega_q - \omega_{k,1}) \Big], \end{aligned} \tag{83} $$

which is not zero (violation of the lepton charge conservation in the decay vertex), in contradiction with the prediction of the weak interaction Hamiltonian. Thus, with same logic used in the QFT formalism, also the Pontecorvo formalism should be claimed to be inconsistent. However, the result (83) only reflects the fact that the adiabatic hypothesis requires much care if used in the study of mixed neutrinos since these are not LSZ states [30, 31] .

As $|V_{\mathbf{k}}| \to 0$, $|U_{\mathbf{k}}| \to 1$, the Pontecorvo result Eq.(83) and the QFT result Eq.(77) are recognized to coincide. Both of them go to zero as $u^r_{\mathbf{k},1} \simeq u^r_{\mathbf{k},2}$, $\omega_{k,1} \simeq \omega_{k,2}$, thus recovering agreement with expected result on the basis of weak interaction Hamiltonian.

C. Amplitude of weak interaction processes with flavor states of neutrinos produced in decay processes

In the present Section, we study the amplitudes for the above processes 1. and 2. by using a further representation of the flavor states, e.g. the one introduced in Ref.[34], where the details of the production mechanism were taken into account for the definition of the flavor states.

The calculation of the amplitudes $A_{W^+\to e^+ + \nu_\mu}$, and $A_{W^+\to e^+ + \nu_e}$, by using the representation of Ref.[34], gives a result which is different from the one of Pontecorvo, unless one goes to the relativistic limit. To see this, let us consider a charged current weak interaction process:

$$P_i \to P_f + l^+_\sigma + \nu_\sigma, \tag{84}$$

where P_i, P_f are initial and final particles, and l^+_σ is the charged lepton associated to the flavor neutrino state ν_σ, $\sigma = e, \mu$. The flavor neutrino created in the decay (84) is now assumed [34] to be represented by the normalized flavor state

$$|\nu_\sigma\rangle = \frac{1}{\sqrt{\sum_i |A_{\sigma i}|^2}} \sum_i A_{\sigma i} |\nu_i\rangle, \tag{85}$$

which is a superposition of massive neutrino states $|\nu_i\rangle$, $i = 1, 2$. Such states, like the Pontecorvo states, are not eigenstates of the flavor charges (38). In Eq.(85), $A_{\sigma i}$ is the amplitude of production of ν_i that depends on the production process, given by

$$A_{\sigma i} = \langle \nu_i, l^+_\sigma, P_f |\, S \,|P_i\rangle\,, \tag{86}$$

where S is the S-matrix operator. The amplitude of the process (84) is then

$$A_{P_i \to P_f l^+_\sigma \nu_\sigma} = \frac{1}{\sqrt{\sum_i |A_{\sigma i}|^2}} \sum_i A^*_{\sigma i} \langle \nu_i, l^+_\sigma, P_f |\, S \,|P_i\rangle = \sqrt{\sum_i |A_{\sigma i}|^2}\,. \tag{87}$$

The weak interaction Hamiltonian is

$$H_{int}(x) = -\frac{G_F}{\sqrt{2}}\, j^\dagger_\mu(x)\, j^\mu(x)\,, \tag{88}$$

where G_F is the Fermi constant and $j^\mu(x)$ is the weak charged current, given by

$$j^\mu(x) = \sum_{\sigma=e,\mu} \overline{\nu}_\sigma(x)\, \gamma^\mu(1-\gamma^5)\, l_\sigma(x) \;+\; h^\mu(x)\,, \tag{89}$$

with $h^\mu(x)$ the hadronic weak charged current. Being the flavor neutrino fields

$$\nu_\sigma = \sum_i U^*_{\sigma i}\, \nu_i \tag{90}$$

with U the unitary mixing matrix, $j^\mu(x)$ can be written as

$$j^\mu(x) = \sum_{\sigma=e,\mu} \sum_i U^*_{\sigma i}\, \overline{\nu}_i(x)\, \gamma^\mu(1-\gamma^5)\, l_\sigma(x) \;+\; h^\mu(x)\,. \tag{91}$$

The amplitude $A_{\sigma i}$ (86) is then

$$A_{\sigma i} = U^*_{\sigma i}\, M_{\sigma i}\,, \tag{92}$$

where

$$M_{\sigma i} = -\frac{i\,G_F}{\sqrt{2}} \int d^4x \, \langle \nu_i, l_\sigma^+ | \left[\overline{\nu}_i(x)\, \gamma^\mu (1-\gamma^5)\, l_\sigma(x) \right] |0\rangle \, J_\mu^{P_i \to P_f}(x)\,, \tag{93}$$

with $J_\mu^{P_i \to P_f}(x)$ the matrix element of the $P_i \to P_f$ transition. The amplitude of the process (84) is then given by

$$A_{P_i \to P_f l_\sigma^+ \nu_\sigma} = \sqrt{\sum_i |A_{\sigma i}|^2} = \sqrt{\sum_i |U_{\sigma i}|^2 \, |M_{\sigma i}|^2}\,. \tag{94}$$

Let us now consider $W^+ \to e^+ + \nu_e$ and $W^+ \to e^+ + \nu_\mu$.

1. *Decay* $W^+ \to e^+ + \nu_e$

The Hamiltonian is written as

$$H_{int}(x) = \frac{g}{2\sqrt{2}} W_\mu^+(x) \sum_i U_{\sigma i}^* \, \overline{\nu}_i(x)\, \gamma^\mu (1-\gamma^5) e(x). \tag{95}$$

The flavor neutrino $|\nu_e\rangle$ created in the decay is assumed to be given by Eq.(85) where $\sigma = e$. $A_{e\,j}$ is then computed as

$$A_{e\,j} = \langle \nu_j, e^+ | \left[-i \int d^4x \, H_{int}(x) \right] |W_{\mathbf{p},\lambda}^+\rangle = U_{e\,j}^* \, M_{e\,j} \tag{96}$$

and

$$M_{e\,j} \;=\; \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}} \, \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \, \overline{u}_{\mathbf{k},j}^r \, \gamma^\mu (1-\gamma^5)\, v_{\mathbf{q},e}^s \;\; \delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}) \, \delta(\omega_p - \omega_q - \omega_{k,j}), \qquad j = 1, 2. \tag{97}$$

The amplitude $A_{W^+ \to e^+ + \nu_e}$ is then

$$A_{W^+ \to e^+ + \nu_e} = \sqrt{\sum_i |U_{e\,i}|^2 \, |M_{e\,i}|^2} = \sqrt{\cos^2\theta \, |M_{e\,1}|^2 + \sin^2\theta \, |M_{e\,2}|^2}\,, \tag{98}$$

which is different from the Pontecorvo result (cf. Eq.(80)).

Since the experiment is not sensitive to the dependence of $M_{e\,j}$ on the different neutrino masses, it is possible to approximate $M_{e\,i} \simeq M_i$. Since $\sum_i |U_{e\,i}|^2 = 1$, we have $A_{W^+ \to e^+ + \nu_e} = M_i$ that coincides with the standard decay amplitude for massless neutrinos.

2. *Decay* $W^+ \to e^+ + \nu_\mu$

The flavor neutrino $|\nu_\mu\rangle$ is created in the decay $W^+ \to \mu^+ + \nu_\mu$. The Hamiltonian responsible of this decay is now

$$H_{int}(x) = -\frac{g}{2\sqrt{2}} W_\alpha^+(x)\, \overline{\nu}_\mu(x)\, \gamma^\alpha (1-\gamma^5)\, \mu(x), \tag{99}$$

and $|\nu_\mu\rangle$ is assumed to be given by Eq.(85) where $\sigma = \mu$. $A_{\mu\,j}$ is then found to be

$$A_{\mu\,j} \;=\; \langle \nu_j, \mu^+ | \left[i \int d^4x \, \frac{g}{2\sqrt{2}} W_\alpha^+(x) \overline{\nu}_\mu(x) \, \gamma^\alpha (1-\gamma^5)\, \mu(x) \right] |W_{\mathbf{p},\lambda}^+\rangle = U_{\mu\,j}^* \, M_{\mu\,j} \tag{100}$$

and

$$M_{\mu\,j} \;=\; \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}} \, \frac{\varepsilon_{\mathbf{p},\alpha,\lambda}}{\sqrt{2\omega_p}} \, \overline{u}_{\mathbf{k},j}^r \, \gamma^\alpha (1-\gamma^5)\, v_{\mathbf{q}',\mu}^s \;\; \delta^3(\mathbf{p}-\mathbf{q}'-\mathbf{k}) \, \delta(\omega_p - \omega_{q'} - \omega_{k,j}), \qquad j = 1, 2. \tag{101}$$

where $\mathbf{q}'$ is the muon momentum and we have considered the relation $\nu_\mu = \sum_i U^*_{\mu i}\,\nu_i$. The amplitude $A_{W^+\to e^+ + \nu_\mu}$ is then

$$\begin{aligned} A_{W^+\to e^+ + \nu_\mu} &= \langle \nu_\mu, e^+ | \left[i \int d^4x \, \frac{g}{2\sqrt{2}} W^+_\alpha(x) \overline{\nu}_e(x)\, \gamma^\alpha (1-\gamma^5)\, e(x) \right] |W^+_{\mathbf{p},\lambda}\rangle \\ &= \frac{1}{\sqrt{\sum_i |A_{\mu\, i}|^2}} \sum_i A^*_{\mu\, i}\, A_{e\, i} = \frac{1}{\sqrt{\sum_i |U_{\mu\, i}|^2 |M_{\mu\, i}|^2}} \sum_i U_{\mu\, i}\, M^*_{\mu\, i} U^*_{e\, i}\, M_{e\, i} \\ &= \frac{\sin\theta \cos\theta}{\sqrt{\sin^2\theta\, |M_{\mu\, 1}|^2 + \cos^2\theta\, |M_{\mu\, 2}|^2}} \left[-M^*_{\mu\, 1} M_{e\, 1} + M^*_{\mu\, 2} M_{e\, 2} \right] , \end{aligned} \tag{102}$$

which is not zero, thus violating the lepton charge conservation predicted by weak interaction.

Again, as far as the experiment is not sensitive to the dependence of $M_{\sigma\, j}$ ($\sigma = e, \mu$) on the different neutrino masses, $M_{\sigma\, i} \simeq M_i$ and we obtain $A_{W^+\to e^+ + \nu_\mu} = 0$.

V. CONCLUSIONS

In the first part of the paper we have discussed the definition of flavor charges and states for mixed neutrinos in QFT. We did this by analyzing the Lagrangian for the charged weak interaction processes for mixed (Dirac) neutrinos in the case of two generations. The flavor states are then defined as eigenstates of the flavor charges, so they give a correct representation for the neutrino production/detection processes. We have also shown that the usual Pontecorvo states are not eigenstates of the flavor charges. This implies that their use to describe the neutrino production/detection in charged current weak interaction processes produces a violation of lepton charge conservation in the vertex and originates inconsistency of measured events with the Pontecorvo oscillation formulas.

In the second part of the paper, we have considered the related problem of the computation of the weak interaction processes $W^+ \to e^+ + \nu_e$ and $W^+ \to e^+ + \nu_\mu$, in the Pontecorvo formalism and in the QFT formalism. The results in both the formalisms have been shown to coincide in the relativistic limit or in the limit of equal mass neutrinos. Using the adiabatic hypothesis, by which time integrations are taken from $t = -\infty$ to $t = +\infty$, leads to results contrasting with the form of the weak interaction Hamiltonian since oscillating neutrinos cannot be treated [30] as usual LSZ fields in QFT [31]. The contradiction is absent in the realistic limit where neutrino masses can be neglected.

We finally note that, as already stressed in Ref. [30], the issue of the possibility of using different basis in which mixed neutrino fields can be expanded is also present for a free field in QFT. The basis to be used is fixed by resorting to the experimental values of the masses. In the case of mixed neutrinos, we have that $[Q_{\nu_\sigma}(t)\, ,\, H] \neq 0$. Therefore, one has two alternative options: to work with mass eigenstates or with (lepton) charge eigenstates. In both cases the expansion for the field is fixed unambiguously by the experiment. It is a well known and peculiar feature of QFT the one of dealing with the freedom in the choice of the state expansion basis: the observed values of the masses and charges are introduced "by hand" in the renormalization process [31]. In QFT it is also a standard matter to work with equal time constraints in computing field theory quantities (e.g. equal time commutators). Therefore, there are no reasons to put forward objections [33, 37] concerning these specific features of the QFT mixing formalism.

APPENDIX A: THE VACUUM STRUCTURE FOR FERMION MIXING

The general frame of the QFT formalism of the neutrino mixing is summarized as follows. For a detailed review see [23]. We consider the Pontecorvo mixing relations

$$\begin{aligned} \nu_e(x) &= \nu_1(x) \cos\theta + \nu_2(x) \sin\theta \\ \nu_\mu(x) &= -\nu_1(x) \sin\theta + \nu_2(x) \cos\theta \end{aligned} \tag{A1}$$

where $\nu_e(x)$ and $\nu_\mu(x)$ are the Dirac neutrino fields with definite flavors. $\nu_1(x)$ and $\nu_2(x)$ are the free neutrino fields with definite masses m_1 and m_2, respectively. The fields $\nu_1(x)$ and $\nu_2(x)$ are written as

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},i}\, \alpha^r_{\mathbf{k},i}(t) + v^r_{-\mathbf{k},i}\, \beta^{r\dagger}_{-\mathbf{k},i}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad i = 1, 2 \tag{A2}$$

with $\alpha^r_{\mathbf{k},i}(t) = \alpha^r_{\mathbf{k},i}\, e^{-i\omega_{k,i} t}$, $\beta^{r\dagger}_{\mathbf{k},i}(t) = \beta^{r\dagger}_{\mathbf{k},i}\, e^{i\omega_{k,i} t}$, and $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. The operator $\alpha^r_{\mathbf{k},i}$ and $\beta^r_{\mathbf{k},i}$, $i = 1, 2$, $r = 1, 2$ are the annihilator operators for the vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha^r_{\mathbf{k},i}|0\rangle_{12} = \beta^r_{\mathbf{k},i}|0\rangle_{12} = 0$. The anticommutation relations are: $\left\{\nu_i^{\alpha}(x), \nu_j^{\beta\dagger}(y)\right\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}$, with $\alpha, \beta = 1, ...4$, and $\left\{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\right\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}$; $\left\{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\right\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}$, with $i,j = 1,2$. All other anticommutators are zero. The orthonormality and completeness relations are: $u^{r\dagger}_{\mathbf{k},i}u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i}v^s_{\mathbf{k},i} = \delta_{rs}$, $u^{r\dagger}_{\mathbf{k},i}v^s_{-\mathbf{k},i} = v^{r\dagger}_{-\mathbf{k},i}u^s_{\mathbf{k},i} = 0$, and $\sum_r(u^r_{\mathbf{k},i}u^{r\dagger}_{\mathbf{k},i} + v^r_{-\mathbf{k},i}v^{r\dagger}_{-\mathbf{k},i}) = 1$.

We construct the generator for the mixing transformation Eqs.(A1) and define [15]:

$$\nu_e^{\alpha}(x) = G^{-1}_\theta(t)\; \nu_1^{\alpha}(x)\; G_\theta(t) \tag{A3}$$

$$\nu_\mu^{\alpha}(x) = G^{-1}_\theta(t)\; \nu_2^{\alpha}(x)\; G_\theta(t)$$

where $G_\theta(t)$ is given by

$$G_\theta(t) = exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right] \,, \tag{A4}$$

and is, at finite volume, an unitary operator, $G^{-1}_\theta(t) = G_{-\theta}(t) = G^\dagger_\theta(t)$, preserving the canonical anticommutation relations. The generator $G^{-1}_\theta(t)$ maps the Hilbert spaces for free fields $\mathcal{H}_{1,2}$ to the Hilbert spaces for interacting fields $\mathcal{H}_{e,\mu}$: $G^{-1}_\theta(t) : \mathcal{H}_{1,2} \mapsto \mathcal{H}_{e,\mu}$. In particular for the vacuum $|0\rangle_{1,2}$ we have, at finite volume V:

$$|0(t)\rangle_{e,\mu} = G^{-1}_\theta(t)\; |0\rangle_{1,2}\;. \tag{A5}$$

$|0\rangle_{e,\mu}$ is the vacuum for $\mathcal{H}_{e,\mu}$, which we will refer to as the flavor vacuum. The explicit expression for $|0\rangle_{e,\mu}$ at time $t=0$ in the reference frame for which $\mathbf{k} = (0,0,|\mathbf{k}|)$ is

$$|0\rangle^{\mathbf{k}}_{e,\mu} \;=\; \prod_r \Big[(1-\sin^2\theta\; |V_\mathbf{k}|^2) - \epsilon^r \sin\theta\; \cos\theta\; |V_\mathbf{k}|(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}) + \tag{A6}$$

$$+\;\; \epsilon^r \sin^2\theta\; |V_\mathbf{k}||U_\mathbf{k}|(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\; |V_\mathbf{k}|^2 \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]|0\rangle_{1,2} \tag{A7}$$

Due to the linearity of $G_\theta(t)$, we can define the flavor annihilators, relative to the fields $\nu_e(x)$ and $\nu_\mu(x)$ at each time expressed as (we use $(\sigma, i) = (e,1), (\mu, 2)$):

$$\alpha^r_{\mathbf{k},\sigma}(t) \;\equiv\; G^{-1}_\theta(t)\; \alpha^r_{\mathbf{k},i}(t)\; G_\theta(t),$$

$$\beta^r_{\mathbf{k},\sigma}(t) \;\equiv\; G^{-1}_\theta(t)\; \beta^r_{\mathbf{k},i}(t)\; G_\theta(t). \tag{A8}$$

The flavor fields are then rewritten into the form:

$$\nu_\sigma(\mathbf{x},t) \;=\; \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r} e^{i\mathbf{k}.\mathbf{x}} \left[u^r_{\mathbf{k},i}\alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\right], \tag{A9}$$

i.e. they can be expanded in the same bases as ν_i.

The flavor annihilation operators can be calculated explicitly and in the reference frame such that $\mathbf{k} = (0,0,|\mathbf{k}|)$ we have

$$\begin{aligned}
\alpha^r_{\mathbf{k},e}(t) &= \cos\theta\; \alpha^r_{\mathbf{k},1}(t) \;+\; \sin\theta\; \Big(|U_\mathbf{k}|\; \alpha^r_{\mathbf{k},2}(t) \;+\; \epsilon^r\; |V_\mathbf{k}|\; \beta^{r\dagger}_{-\mathbf{k},2}(t)\Big) \\
\alpha^r_{\mathbf{k},\mu}(t) &= \cos\theta\; \alpha^r_{\mathbf{k},2}(t) \;-\; \sin\theta\; \Big(|U_\mathbf{k}|\; \alpha^r_{\mathbf{k},1}(t) \;-\; \epsilon^r\; |V_\mathbf{k}|\; \beta^{r\dagger}_{-\mathbf{k},1}(t)\Big) \\
\beta^r_{-\mathbf{k},e}(t) &= \cos\theta\; \beta^r_{-\mathbf{k},1}(t) \;+\; \sin\theta\; \Big(|U_\mathbf{k}|\; \beta^r_{-\mathbf{k},2}(t) \;-\; \epsilon^r\; |V_\mathbf{k}|\; \alpha^{r\dagger}_{\mathbf{k},2}(t)\Big) \\
\beta^r_{-\mathbf{k},\mu}(t) &= \cos\theta\; \beta^r_{-\mathbf{k},2}(t) \;-\; \sin\theta\; \Big(|U_\mathbf{k}|\; \beta^r_{-\mathbf{k},1}(t) \;+\; \epsilon^r\; |V_\mathbf{k}|\; \alpha^{r\dagger}_{\mathbf{k},1}(t)\Big),
\end{aligned} \tag{A10}$$

with $\epsilon^r = (-1)^r$ and

$$|U_\mathbf{k}| \;\equiv\; u^{r\dagger}_{\mathbf{k},i}u^r_{\mathbf{k},j} = v^{r\dagger}_{-\mathbf{k},i}v^r_{-\mathbf{k},j}$$

$$|V_\mathbf{k}| \;\equiv\; \epsilon^r\; u^{r\dagger}_{\mathbf{k},1}v^r_{-\mathbf{k},2} = -\epsilon^r\; u^{r\dagger}_{\mathbf{k},2}v^r_{-\mathbf{k},1} \tag{A11}$$

with $i,j = 1,2$ and $i \neq j$. We have:

$$|U_{\mathbf{k}}| = \left(\frac{\omega_{k,1}+m_1}{2\omega_{k,1}}\right)^{\frac{1}{2}} \left(\frac{\omega_{k,2}+m_2}{2\omega_{k,2}}\right)^{\frac{1}{2}} \left(1+\frac{\mathbf{k}^2}{(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}\right)$$
$$|V_{\mathbf{k}}| = \left(\frac{\omega_{k,1}+m_1}{2\omega_{k,1}}\right)^{\frac{1}{2}} \left(\frac{\omega_{k,2}+m_2}{2\omega_{k,2}}\right)^{\frac{1}{2}} \left(\frac{|\mathbf{k}|}{(\omega_{k,2}+m_2)}-\frac{|\mathbf{k}|}{(\omega_{k,1}+m_1)}\right) \tag{A12}$$

$$|U_{\mathbf{k}}|^2+|V_{\mathbf{k}}|^2 = 1. \tag{A13}$$

Note that the following relations hold:

$$\overline{u}^r_{\mathbf{k},1}\sum_s u^{r\dagger}_{\mathbf{k},1}u^s_{\mathbf{k},2}+\overline{v}^r_{-\mathbf{k},1}\sum_s v^{r\dagger}_{-\mathbf{k},1}u^s_{\mathbf{k},2} \;=\; \overline{u}^r_{\mathbf{k},2} \tag{A14}$$
$$\overline{u}^r_{\mathbf{k},1}\sum_s u^{r\dagger}_{\mathbf{k},1}v^s_{-\mathbf{k},2}+\overline{v}^r_{-\mathbf{k},1}\sum_s v^{r\dagger}_{-\mathbf{k},1}v^s_{-\mathbf{k},2} \;=\; \overline{v}^r_{-\mathbf{k},2}\,, \tag{A15}$$

and in the reference frame for which $\mathbf{k}=(0,0,|\mathbf{k}|)$ we have

$$\overline{u}^r_{\mathbf{k},1}\,|U_{\mathbf{k}}|-\varepsilon^r\,\overline{v}^r_{-\mathbf{k},1}|V_{\mathbf{k}}| \;=\; \overline{u}^r_{\mathbf{k},2} \tag{A16}$$
$$\overline{u}^r_{\mathbf{k},1}\,|V_{\mathbf{k}}|+\varepsilon^r\,\overline{v}^r_{-\mathbf{k},1}|U_{\mathbf{k}}| \;=\; \varepsilon^r\,\overline{v}^r_{-\mathbf{k},2}\,. \tag{A17}$$

The condensation density is given by

$$_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}|0\rangle_{e,\mu} \;=\; {}_{e,\mu}\langle 0|\beta^{r\dagger}_{\mathbf{k},i}\beta^r_{\mathbf{k},i}|0\rangle_{e,\mu} \;=\; \sin^2\theta\;|V_{\mathbf{k}}|^2\;, \qquad i=1,2\,. \tag{A18}$$

The explicit expressions for the flavor states $|\nu^r_{\mathbf{k},e}\rangle$ and $|\nu^r_{\mathbf{k},\mu}\rangle$ at time $t=0$, in the reference frame for which $\mathbf{k}=(0,0,|\mathbf{k}|)$ are

$$|\nu^r_{\mathbf{k},e}\rangle \;\equiv\; \alpha^{r\dagger}_{\mathbf{k},e}(0)|0\rangle_{e,\mu} = \Big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1}+|U_{\mathbf{k}}|\;\sin\theta\;\alpha^{r\dagger}_{\mathbf{k},2}-\epsilon^r\;|V_{\mathbf{k}}|\;\sin\theta\;\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\;G^{-1}_{\mathbf{k},s\neq r}(\theta)\prod_{\mathbf{p}\neq\mathbf{k}}G^{-1}_{\mathbf{p}}(\theta)|0\rangle_{1,2}, \tag{A19}$$
$$|\nu^r_{\mathbf{k},\mu}\rangle \;\equiv\; \alpha^{r\dagger}_{\mathbf{k},\mu}(0)|0\rangle_{e,\mu} = \Big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},2}-|U_{\mathbf{k}}|\;\sin\theta\;\alpha^{r\dagger}_{\mathbf{k},1}+\epsilon^r\;|V_{\mathbf{k}}|\;\sin\theta\;\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\Big]\;G^{-1}_{\mathbf{k},s\neq r}(\theta)\prod_{\mathbf{p}\neq\mathbf{k}}G^{-1}_{\mathbf{p}}(\theta)|0\rangle_{1,2}\,, \tag{A20}$$

where $G(\theta,t)=\prod_{\mathbf{p}}\prod_{s=1}^2 G_{\mathbf{p},s}(\theta,t)$. In these states a multiparticle component is present, disappearing in the relativistic limit $|\mathbf{k}|\gg\sqrt{m_1m_2}$: in this limit, since $|U_{\mathbf{k}}|^2\longrightarrow 1$ and $|V_{\mathbf{k}}|^2\longrightarrow 0$, the (quantum-mechanical) Pontecorvo states are recovered.

The flavor oscillation formulas are derived by computing, in the Heisenberg representation, the expectation value of the flavor charge operators on the flavor state. We have

$$_{e,\mu}\langle 0|::Q_{\nu_e}(t)::|0\rangle_{e,\mu} ={}_{e,\mu}\langle 0|Q_{\nu_\mu}(t)|0\rangle_{e,\mu}=0, \tag{A21}$$

and [16]:

$$\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t) \;=\; \langle\nu^r_{\mathbf{k},e}|::Q_{\nu_e}(t)::|\nu^r_{\mathbf{k},e}\rangle = 1-\sin^2(2\theta)\Big[\,|U_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right)+|V_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\Big], \tag{A22}$$
$$\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_\mu}(t) \;=\; \langle\nu^r_{\mathbf{k},e}|::Q_{\nu_\mu}(t)::|\nu^r_{\mathbf{k},e}\rangle = \sin^2(2\theta)\Big[\,|U_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right)+|V_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\Big]. \tag{A23}$$

The charge conservation is ensured at any time:

$$\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t)+\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_\mu}(t)=1. \tag{A24}$$

The differences with respect to the Pontecorvo formulas are: the energy dependence of the amplitudes, and the additional oscillating term. In the relativistic limit: $|\mathbf{k}| \gg \sqrt{m_1 m_2}$, we have $|U_{\mathbf{k}}|^2 \longrightarrow 1$ and $|V_{\mathbf{k}}|^2 \longrightarrow 0$ and the traditional formulas are recovered.

Similar results are obtained for three flavor neutrino fields [22] and for boson fields [20, 24].

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